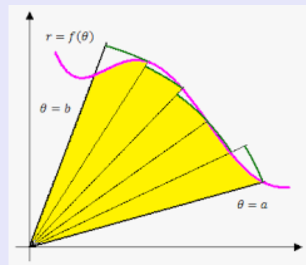
Calculus II

Lecture 14



Feb 19-8:47 AM

Class QZ 9

Find $\displaystyle\int \frac{2}{x^2-4x+3}\,dx = \int \left[\frac{1}{x-3} - \frac{1}{x-1}\right]dx$

$= \ln|x-3| - \ln|x-1| + C$

$= \boxed{\ln\left|\frac{x-3}{x-1}\right| + C}$

$$\frac{2}{x^2-4x+3} = \frac{2}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}$$

$2 = A(x-1) + B(x-3)$

$x=1 \rightarrow B=-1$ ✓

$x=3 \rightarrow A=1$ ✓

$\rightarrow x^2-4x+3 = x^2-4x+4-4+3$

$= x^2-4x+4-1$

$= (x-2)^2 - 1^2$

$\displaystyle\int \frac{1}{u^2-a^2}\,du =$

Jul 2-10:51 AM

Class QZ 10 (15 pts)

Evaluate $\int_1^{\infty} \frac{1}{x^2+x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+x}\,dx$

$$\int \frac{1}{x^2+x}\,dx = \int \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \ln|x| - \ln|x+1| + C$$

$$= \ln\left|\frac{x}{x+1}\right| + C$$

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$1 = A(x+1) + Bx$

$1 = Ax + A + Bx$

$1 = (A+B)x + A$

$A+B=0 \qquad 1+B=0$

$A=1 \qquad B=-1$

$$= \lim_{t\to\infty} \ln\left|\left(\frac{t}{t+1}\right)\right| - \ln\left|\frac{1}{2}\right|, \quad \frac{t}{t+1} \to 1$$

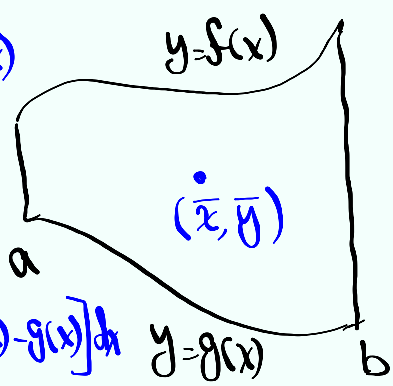
$$= \cancel{\ln 1}^{\,\to 0} - \ln\frac{1}{2}$$

$$= -\left[\cancel{\ln 1}^{\,\to 0} - \ln 2\right] = \boxed{\ln 2}$$

Convergent

Centroid of a Region $(\bar{x}, \bar{y})$

$f(x) \geq g(x)$



$$A = \int_a^b [f(x) - g(x)]\,dx$$

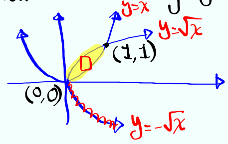
1) Find its Area A

2) $\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$

3) $\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right]dx$

Consider the region bounded by $y=x^2$ & $x=y^2$. $y=\sqrt{x}$

1) Draw the region



2) Find its area.

$$A=\int_0^1\left[\sqrt{x}-x^2\right]dx=\left(\frac{x^{3/2}}{3/2}-\frac{x^3}{3}\right)\Big|_0^1=\frac{2}{3}-\frac{1}{3}=\boxed{\frac{1}{3}}$$

3) Find its Centroid $(\bar{x},\bar{y})$

$$\bar{x}=\frac{1}{A}\int_0^1 x\Big[\underset{\text{Top}}{\sqrt{x}}-\underset{\text{Bottom}}{x^2}\Big]dx=\frac{1}{\frac{1}{3}}\int_0^1\left(x^{\frac{3}{2}}-x^3\right)dx$$

$$=3\left[\frac{x^{5/2}}{5/2}-\frac{x^4}{4}\right]\Big|_0^1=3\left(\frac{2}{5}-\frac{1}{4}\right)=3\left(\frac{8-5}{20}\right)=\boxed{\frac{9}{20}}$$

$$\bar{y}=\frac{1}{A}\int_0^1\frac{1}{2}\left[[f(x)]^2-(g(x))^2\right]dx$$

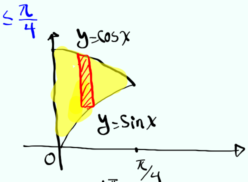
$$=\frac{1}{\frac{1}{3}}\cdot\frac{1}{2}\int_0^1\left[(\sqrt{x})^2-(x^2)^2\right]dx=\frac{3}{2}\left[\frac{x^2}{2}-\frac{x^5}{5}\right]\Big|_0^1=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{3}{2}\cdot\frac{3}{10}=\boxed{\frac{9}{20}}$$

$$(\bar{x},\bar{y})=\left(\frac{9}{20},\frac{9}{20}\right)$$

Jul 3-8:26 AM

Consider the region bounded by $y=\sin x$, $y=\cos x$ for $0\le x\le\frac{\pi}{4}$

1) Draw the region



2) Find its area.

$$A=\int_0^{\pi/4}\left[\cos x-\sin x\right]dx=\left(\sin x+\cos x\right)\Big|_0^{\pi/4}=\boxed{\sqrt{2}-1}$$

3) Find its Centroid $(\bar{x},\bar{y})$

$$\bar{x}=\frac{1}{A}\int_a^b x\left[f(x)-g(x)\right]dx=\frac{1}{\sqrt{2}-1}\int_0^{\pi/4}x\left[\cos x-\sin x\right]dx$$

$$=\frac{1}{\sqrt{2}-1}\int_0^{\pi/4}(x\cos x-x\sin x)\,dx$$

$$=\frac{1}{\sqrt{2}-1}\left[x\sin x+\cos x+x\cos x-\sin x\right]\Big|_0^{\pi/4}$$

$$=\frac{1}{\sqrt{2}-1}\left[\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}+\cancel{\frac{\sqrt{2}}{2}}+\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}-\cancel{\frac{\sqrt{2}}{2}}-1\right]$$

$$=\frac{1}{\sqrt{2}-1}\left[\frac{\pi\sqrt{2}}{4}-1\right]=\boxed{\frac{\pi\sqrt{2}-4}{4(\sqrt{2}-1)}}$$

$\int x\cos x\,dx$

$u=x$ $dv=\cos x\,dx$

$du=dx$ $v=\sin x$

$=x\sin x-\int\sin x\,dx$

$=x\sin x+\cos x$

$\int x\sin x\,dx=$

$u=x$ $dv=\sin x\,dx$

$du=dx$ $v=-\cos x$

$-x\cos x+\int\cos x\,dx=$

$-x\cos x+\sin x$

Jul 3-8:39 AM

$$\bar{y} = \frac{1}{A}\int_0^{\pi/4} \frac{1}{2}\left[[f(x)]^2 - [g(x)]^2 \right] dx$$

$$= \frac{1}{\sqrt{2}-1} \cdot \frac{1}{2}\int_0^{\pi/4} [\underbrace{\cos^2 x - \sin^2 x}_{\cos 2x}]\, dx$$

$$= \frac{1}{2(\sqrt{2}-1)} \cdot \int_0^{\pi/4} \cos 2x \, dx$$

$$= \frac{1}{2(\sqrt{2}-1)} \cdot \frac{1}{2} \sin 2x \Big|_0^{\pi/4} = \frac{1}{4(\sqrt{2}-1)}\left[\overset{1}{\cancel{\sin \tfrac{\pi}{2}}} - \overset{0}{\cancel{\sin 0}}\right]$$

$$= \frac{1}{4(\sqrt{2}-1)}$$

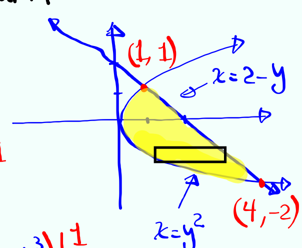
$$(\bar{x}, \bar{y}) = \left(\frac{\pi\sqrt{2} - 4}{4(\sqrt{2}-1)} , \frac{1}{4(\sqrt{2}-1)} \right)$$

Jul 3-8:54 AM

Consider the region bounded by

$x + y = 2$ and $x = y^2$

1) Draw the region. Clearly label all intersection Points.



$x + y = 2$
$y^2 + y = 2$

$\to y^2 + y - 2 = 0$
$(y+2)(y-1) = 0$
$y = -2,\ y = 1$

2) Find its area.

$$A = \int_{-2}^{1} [2 - y - y^2]\, dy = \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} - \frac{-8}{3}\right)$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - \frac{8}{3}$$

$$= 8 - \frac{1}{2} - \frac{9}{3} = 8 - \frac{1}{2} - 3 = 5 - \frac{1}{2}$$

$$= 4\frac{1}{2}$$

$$= \boxed{\frac{9}{2}}$$

3) Find its Centroid $(\bar{x}, \bar{y})$.

Jul 3-9:02 AM

Since it is easier to do integration with respect to y,

$$\bar{y} = \frac{1}{A}\int_{-2}^{1} y[f(y) - g(y)]\,dy \quad \&$$

$$\bar{x} = \frac{1}{A}\int_{-2}^{1} \frac{1}{2}\left[(f(y))^2 - (g(y))^2\right]dy$$

$$\bar{y} = \frac{2}{9}\int_{-2}^{1} y[2 - y - y^2]\,dy = \frac{2}{9}\left[y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]\Bigg|_{-2}^{1}$$

$$= \frac{2}{9}\left[\left(1 - \frac{1}{3} - \frac{1}{4}\right) - \left(\cancel{4} + \frac{8}{3} - \cancel{4}\right)\right]$$

$$= \frac{2}{9}\left[1 - 3 - \frac{1}{4}\right] = \frac{2}{9}\left[-2 - \frac{1}{4}\right] = \frac{2}{9}\cdot\frac{-9}{4} = \boxed{\frac{-1}{2}}$$

$$\bar{x} = \frac{1}{A}\int_{-2}^{1} \frac{1}{2}\left[(f(y))^2 - (g(y))^2\right]dy$$

$$= \frac{2}{9}\cdot\frac{1}{2}\int_{-2}^{1}\left[(2-y)^2 - (y^2)^2\right]dy$$

$$= \frac{1}{9}\int_{-2}^{1}\left[4 - 4y + y^2 - y^4\right]dy$$

$$= \frac{1}{9}\left[4y - 2y^2 + \frac{y^3}{3} - \frac{y^5}{5}\right]\Bigg|_{-2}^{1}$$

$$= \frac{1}{9}\left[\left(4 - 2 + \frac{1}{3} - \frac{1}{5}\right) - \left(-8 - 8 - \frac{8}{3} + \frac{32}{5}\right)\right]$$

$$= \frac{1}{9}\left[2 + 16 + 3 - \frac{33}{5}\right] = \frac{1}{9}\left[21 - \frac{33}{5}\right]$$

$$= \frac{1}{9}\cdot\frac{105-33}{5} = \frac{\cancel{72}^{\,8}}{\cancel{9}\cdot 5}$$

$$\boxed{\bar{x} = \frac{8}{5}}$$

$$\boxed{(\bar{x},\bar{y}) = \left(\frac{8}{5}, \frac{-1}{2}\right)}$$

Jul 3-9:40 AM

$$\int \frac{1}{x^2+x}\,dx$$

$$\frac{1}{x^2+x} = \frac{1+x-x}{x^2+x} = \frac{1+x}{x^2+x} - \frac{x}{x^2+x}$$

$$= \frac{\cancel{1+x}}{x\cancel{(x+1)}} - \frac{\cancel{x}}{\cancel{x}(x+1)}$$

$$= \frac{1}{x} - \frac{1}{x+1}$$

Jul 3-9:53 AM

Rotate the region bounded by $f(x) = \sqrt{R^2 - x^2}$ and x-axis by x-axis.

Top half of a circle → $y = \sqrt{R^2 - x^2}$

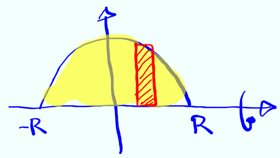
$$y^2 = R^2 - x^2$$

$$x^2 + y^2 = R^2$$

$(0,0)$, Radius R

Find its volume.

R is a constant.



Solid obtained is a Sphere → Radius R

$$\boxed{V = \frac{4\pi R^3}{3}}$$

Disk

$$V = \int_{-R}^{R} \pi \left[\sqrt{R^2 - x^2}\right]^2 dx = 2\int_0^R \pi (R^2 - x^2)\, dx$$

$$= 2\pi \left[R^2 x - \frac{x^3}{3}\right]\Big|_0^R$$

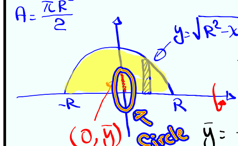
$$= 2\pi \left[R^3 - \frac{R^3}{3}\right] = 2\pi \cdot \frac{2}{3} R^3$$

$$= \frac{4\pi R^3}{3}$$

Jul 3-9:55 AM

Find the centroid of the region.

$A = \frac{\pi R^2}{2}$



$$\bar{y} = \frac{1}{A} \int_{-R}^{R} \frac{1}{2}\left[\left(\sqrt{R^2-x^2}\right)^2 - (0)^2\right] dx$$

$$\bar{y} = \frac{\cancel{2}}{\pi R^2} \cdot \frac{1}{\cancel{2}} \int_{-R}^{R} (R^2 - x^2)\, dx$$

$$\bar{y} = \frac{1}{\pi R^2} \cdot 2 \int_0^R (R^2 - x^2)\, dx = \frac{2}{\pi R^2}\left[R^2 x - \frac{x^3}{3}\right]_0^R$$

$$= \frac{2}{\pi R^2} \cdot \frac{2R^3}{3} = \boxed{\frac{4R}{3\pi}}$$

If we take the centroid and rotate it by x-axis.

1) what shape do we make? Circle

2) How far do we go around? Circum. of the Circle

3) Radius of that circle? $\bar{y}$

$$2\pi \bar{y}$$

Multiply by Area of bounded region

$$2\pi \bar{y} \cdot \frac{\pi R^2}{2}$$

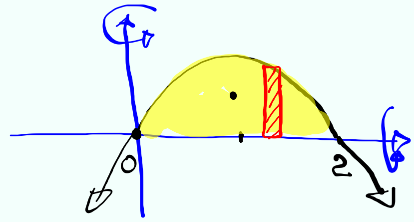
$$= \cancel{2}\cancel{\pi} \cdot \frac{4R}{3\cancel{\pi}} \cdot \frac{\pi R^2}{\cancel{2}}$$

$$= \frac{4\pi R^3}{3}$$ Volume of Same Sphere.

Jul 3-10:03 AM

Consider the region bounded by $f(x) = 2x - x^2$ and x-axis.

Parabola open downward



1) Find its area

$$A = \int_0^2 [2x - x^2]\,dx = \boxed{\frac{4}{3}}$$

2) Find the volume if rotated about x-axis.

Disk $\quad V = \int_0^2 \pi(2x - x^2)^2\,dx = \pi\int_0^2 (4x^2 - 4x^3 + x^4)\,dx = \boxed{\frac{16\pi}{15}}$

3) Find the volume if rotated about y-axis.

Shell $\quad V = \int_0^2 2\pi x(2x - x^2)\,dx = 2\pi\int_0^2 (2x^2 - x^3)\,dx = \boxed{\frac{8\pi}{3}}$

Jul 3-10:15 AM

4) Find its Centroid

$$\bar{x} = \frac{1}{A}\int_0^2 x(2x - x^2)\,dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Bigg|_0^2 = \boxed{1}$$

$$\bar{y} = \frac{1}{A}\int_0^2 \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right]dx$$

See earlier work

$$= \frac{3}{4}\cdot\frac{1}{2}\int_0^2 (2x - x^2)^2\,dx = \frac{3}{8}\cdot\frac{16}{15} = \boxed{\frac{2}{5}}$$

Centroid is $\left(1, \frac{2}{5}\right)$

Take the centroid, go around x-axis, find the distance traveled around.

$$2\pi\bar{y} = 2\pi\cdot\frac{2}{5} = \frac{4\pi}{5}$$

Multiply by the area of the region $\quad \frac{4\pi}{5}\cdot\frac{4}{3} = \frac{16\pi}{15}$

Jul 3-10:26 AM

Take the Centroid go around the y-axis

distance traveled → $2\pi\bar{x} = 2\pi \cdot 1 = 2\pi$

multiply by the area of the region $\quad 2\pi \cdot \frac{4}{3} = \frac{8\pi}{3}$

Summary

If an enclosed region is rotated about a line, totally on one side of it,

(line can not cross the region)

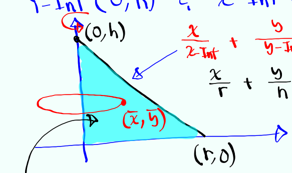
the Volume can be found by

Area of the Region $\cdot$ Distance traveled by its Centroid

Pappus theorem

Jul 3-10:35 AM

Consider the region in QI bounded by x-axis, y-axis, and a line with y-Int $(0,h)$ & x-Int $(r,0)$.



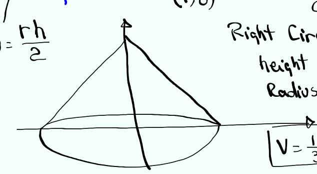
$\frac{x}{x\text{-Int}} + \frac{y}{y\text{-Int}} = 1$

$\frac{x}{r} + \frac{y}{h} = 1$

$\frac{y}{h} = 1 - \frac{x}{r}$

$y = h - \frac{h}{r}x$

$A = \frac{rh}{2}$



Right Circular Cone
height h
Radius r

$\boxed{V = \frac{1}{3}\pi r^2 h}$

V = Area of the region $\cdot$ distance traveled by its Centroid

$V = \frac{rh}{2} \cdot 2\pi \cdot \bar{x}$

$= \frac{rh}{\cancel{2}} \cdot \cancel{2}\pi \cdot \frac{r}{3}$

$= \frac{\pi r^2 h}{3}$ Volume of Cone

$\bar{x} = \frac{1}{A}\int_0^r x\left(h - \frac{h}{r}x\right)dx$

$= \frac{2}{rh} \cdot \left[\frac{hx^2}{2} - \frac{h}{r}\frac{x^3}{3}\right]\Big|_0^r$

$= \frac{2}{rh}\left[\frac{hr^2}{2} - \frac{hr^2}{3}\right]$

$= \frac{\cancel{2}}{\cancel{rh}} \cdot \cancel{h}\cancel{r^2}^{\,r} \cdot \frac{1}{\cancel{6}\,3}$

$= \frac{r}{3}$

Jul 3-10:40 AM

Class QZ 11 (15 pts)

Find the exact value of the surface area if the arc length given by $f(x) = \frac{x^4}{16} + \frac{1}{2x^2}$ for $1 \le x \le 2$ rotated about Y-axis.

$f'(x) = \frac{4x^3}{16} - \frac{1}{x^3} = \frac{x^3}{4} - \frac{1}{x^3}$ $\qquad [f'(x)]^2 = \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6}$

$1 + [f'(x)]^2 = \frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} = \left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2$

$\sqrt{1 + [f'(x)]^2} = \frac{x^3}{4} + \frac{1}{x^3}$

$$S = \int_1^2 2\pi\, x \left(\frac{x^3}{4} + \frac{1}{x^3}\right) dx = 2\pi \int_1^2 \left[\frac{x^4}{4} + \frac{1}{x^2}\right] dx$$

$$= 2\pi \left[\frac{x^5}{20} - \frac{1}{x}\right]\Bigg|_1^2 = 2\pi\left[\left(\frac{32}{20} - \frac{1}{2}\right) - \left(\frac{1}{20} - 1\right)\right] = 2\pi\left[\frac{31}{20} + \frac{1}{2}\right]$$

$$= 2\pi \cdot \frac{41}{20} = \boxed{\frac{41\pi}{10}}$$

Jul 3-10:54 AM